Intuitionistic Implications: On the Logics of Spacetime

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The Main Problem

What is the abstract and the most general notion of implication?

Implication as Internalization

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- Reflexivity, i.e., "$A \vdash A$" for any proposition A. The internalization: $\vdash A \to A$,
- Transitivity, i.e., "$A \vdash B$ *and* $B \vdash C$ *implies* $A \vdash C$" for any propositions A, B, and C. The internalization:

$$(A \to B) \wedge (B \to C) \vdash (A \to C),$$

  for any propositions A, B, and C.

Abstract Implication

Definition

Let $\mathcal{A} = (A, \leq, \wedge, 1)$ be a bounded meet-semilatice. By an implication $\rightarrow: A^{op} \times A \Rightarrow A$ we mean any monotone function with the following properties:

- $a \rightarrow a = 1$,
- $(a \rightarrow b) \wedge (b \rightarrow c) \leq (a \rightarrow c)$,

The structure $(A, \leq, \wedge, 1, \rightarrow)$ is called a strong algebra if $\rightarrow$ is an implication.

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- Let $\mathcal{A}$ be a bounded meet-semilatice. Define $a \rightarrow b = 1$ for all $a, b \in A$.
- Let X be a topological space. Then $U \rightarrow V = int(U^c \cup V)$ over $\mathcal{O}(X)$ is an implication.

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- Gödel's implication on $[0, 1]$ defined by $a \rightarrow b = b$ if $a > b$ and 1 otherwise.

Two Construction Methods

- Let $(A, \leq, \wedge, 1, \to)$ be a strong algebra and $F : A \to A$ be a monotone operation. Define $a \to_F b = F(a) \to F(b)$. Then $\to_F$ is also an implication.

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The first method is the modification factor. However, the applications of the second method on the intuionistic implications play a critical philosophical role. We call these implications *generalized intuitionistic implications.*

Intuitionism: Propositions via Space

Let S be the set of all creative subject's mental states. Then by a proposition P we mean a subset of S consisting of all states in which P holds and this fact is verifiable by finite means. It has three conditions:

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- These subsets are ordered by the partial order $A \vdash B$ that encodes the situation that the truth of A in any state implies the truth of B in the same state.
- *Finite Intersection.* The second structure is the finite meets of the poset, called conjunctions. If both A and B are finitely verifiable propositions, then so is $A \wedge B$. Because, if $A \wedge B$ holds in a state, there are finite verifications for both of them and the combination of these verifications is also finite. Note that the same claim is not necessarily true for infinite conjunctions, because, if the infinite conjunction is true, we need possibly infinite number of verifications that may exceed any possible finite memory.

- *Arbitrary Union.* The last and the third structure is the arbitrary joins called disjunctions. For some set I, if A_i is finitely verifiable for any $i \in I$, then so is $\bigvee_{i \in I} A_i$. Because, if $\bigvee_{i \in I} A_i$ holds in a state, then one of them must hold and since it has a finite verification, the verification also works for the whole disjunction.

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Therefore, it should not be surprising that intuitionistic propositional logic is sound and complete with respect to its topological interpretation that reads a proposition as an open subset of a given topological space. In this sense, intuitionism may be interpreted as the logic of space as opposed to the classical logic that corresponds to the logic of sets or discrete spaces. Compare the set of all opens of a space to the opens of a discrete space, namely the Boolean algebra of all subsets.

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Add the temporal modality, ∇A, meaning "A holds at some point in the past".

- ∇A is a proposition itself. Since, if ∇A holds in a mental state, there is some point in the past in which A holds. But A is a proposition and hence has a finite verification at that point. Therefore, it is easy to bring that verification to the current mental state and save it as some temporal information of the past.

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- ∇ is clearly monotone and union preserving. If $\nabla \bigvee_{i \in I} A_i$ holds at some state, then there *exists* some point in the past in which $\bigvee_{i \in I} A_i$ holds. Hence, one of A_i's must hold in that point which implies ∇A_i holds at the current state. Hence, we have $\bigvee_{i \in I} \nabla A_i$.

Spacetimes

The spatio-temporal structure of the creative subject's mental states is formalized by:

Definition

Let X be a topological space and $\nabla : \mathcal{O}(X) \to \mathcal{O}(X)$ be an increasing and join preserving operation. Then the pair (X, ∇) is called a spacetime.

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Example

For any continuous function $f : X \to X$, the pair (X, f^{-1}) is a spacetime.

Theorem

Let (X, ∇) *be a spacetime. Then there exists an implication* $\to_\nabla$ *on* $\mathcal{O}(X)$ *called generalized intuitionistic implication such that for any* $U, V, W \in \mathcal{O}(X)$ *we have* $\nabla W \cap U \subseteq V$ *iff* $W \subseteq U \to_\nabla V$*, i.e.,* $\nabla(U \to_\nabla V) \cap U \subseteq V$ *and* $U \to_\nabla V$ *is the best such proposition.*

Generalized Intuitionistic Implications

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Proof.

Define $G(U) = \bigcup\{V | \nabla V \subseteq U\}$ and $U \rightarrow_\nabla V$ as $G(int(U^c \cup V))$. It is easy to show that G is meet-preserving. One side of the equivalence is obvious. The other side is the result of join preservability of ∇. Note that $\rightarrow_\nabla$ is the result of the application of the second method on intuitionistic implication on $\mathcal{O}(X)$.

Representation Theorems I

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General Representation Theorem

If $\mathcal{A}$ is a strong algebra then there exists a spacetime (X, ∇) and a meet semi-lattice embedding $i : A \to \mathcal{O}(X)$ and a monotone map $F : \mathcal{O}(X) \to \mathcal{O}(X)$ such that for any $a, b \in A$ we have $i(a \to b) = F(i(a)) \to_{\nabla} F(i(b))$.

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Philosophical Consequence

Any implication is a *generalized intuitionistic implication* up to a modification factor and enlarging the domain of the discourse.

Meet-internalizing Implications

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This is impossible. The reason is that for any spacetime (X, ∇), the implication $\to_\nabla$ has the following *meet-internalizing* property:

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because,

$$\nabla Z \cap U \subseteq V \cap W \text{ iff } Z \subseteq U \to_\nabla V \cap W$$

$$[\nabla Z \cap U \subseteq V \text{ and } \nabla Z \cap U \subseteq W] \text{ iff } [Z \subseteq U \to_\nabla V \text{ and } Z \subseteq U \to_\nabla W]$$

Representation Theorems II

Therefore, the necessary condition for an abstract implication to be embeddable in a spacetime is the meet-internalizing condition. This condition is fortunately sufficient:

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Special Representation Theorem (A., Alizadeh, Memarzadeh)

If $\mathcal{A}$ is a meet internalizing strong algebra, then there exists a spacetime (X, ∇) and a strong algebra embedding $i : \mathcal{A} \to (\mathcal{O}(X), \to_\nabla)$.

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If $\mathcal{A}$ is a meet internalizing strong algebra, then there exists a spacetime (X, ∇) and a strong algebra embedding $i : \mathcal{A} \to (\mathcal{O}(X), \to_\nabla)$.

Philosophical Consequence

Any *reasonable* implication is a *generalized intuitionistic implication*, enlarging the domain of the discourse.

Let $\mathcal{L}_\nabla$ be the usual language of propositional logic with a unary modal operator ∇. Define **STL** as the system consisting of the usual sequent-style rules for all connectives except implication (and hence negation) plus:

Implication Rules:

$$\frac{\Gamma \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, \nabla(A \to B) \Rightarrow C} L\to \qquad \frac{\nabla\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} R\to$$

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Modal Rule:

$$\frac{\Gamma \Rightarrow A}{\nabla\Gamma \Rightarrow \nabla A}\, \nabla$$

Γ includes exactly one formula. If Γ can be arbitrary, the stronger rule is called (N) and the stronger system is **STL**(N).

Topological Semantics

Definition

A topological model is a tuple (X, ∇, V) such that (X, ∇) is a spacetime and $V : \mathcal{L}_\nabla \to \mathcal{O}(X)$ is a valuation function such that: $V(\top) = X$; $V(\bot) = \emptyset$; $V(A \wedge B) = V(A) \cap V(B)$; $V(A \vee B) = V(A) \cup V(B)$; $V(A \to B) = V(A) \to_\nabla V(B)$ and $V(\nabla A) = \nabla V(A)$. We say $(X, \nabla, V) \vDash \Gamma \Rightarrow A$ when $\bigcap_{\gamma \in \Gamma} V(\gamma) \subseteq V(A)$.

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$\Gamma \vdash_{\mathbf{STL}} A$ iff $\Gamma \Rightarrow A$ is valid in all spacetimes.

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$\Gamma \vdash_{\mathbf{STL}} A$ iff $\Gamma \Rightarrow A$ is valid in all spacetimes.

Strong Completeness Theorem

For completeness any fixed discrete space with the cardinality greater than the continuum is sufficient.

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Theorem

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This shows that the logic of spacetime is a refined version of the usual intuitionistic logic.

Thank you for your attention!